Chapter 5 Permutations, Combinations, and Generating Functions

5-1 Permutation and Combination

Rule of sum: The total items can be broken into first and second classes. The first class has m items and the second class has n items. Selecting any one in either class has $m+n$ ways.

Eg. There are 3 men and 4 women in a company. If the boss needs only one person to clean rooms, he has 3+4=7 choices.

Rule of product: A procedure can be broken into first and second stages. If the first stage has m outcomes and the second stage has n outcomes, the total procedure has mn ways.

Eg. There are 3 men and 4 women. If they want to marry to each other, there are couples in 3×4=12 ways.

Permutation: Count arrangements of objects **with reference to order**.

Eg. How many permutations of the letters *ABCD* contain the substring *CD*?

(Sol.) 3!=6 (*ABCD*, *BACD*, *ACDB*, *BCDA*, *CDAB*, *CDBA*)

Eg. How many permutations of the letters *ABCD* contain the letters *CD* together in any order?

(Sol.) 3!2!=12 (*ABCD*, *BACD*, *ACDB*, *BCDA*, *CDAB*, *CDBA*, *ABDC*, *BADC*, *ADCB*, *BDCA*, *DCAB*, *DCBA*)

The number of *r*-permutations of *n* distinct objects: $P_r^n = \dfrac{n!}{(n-r)!}$

Eg. In how many ways can we select a chairman, vice-chairman, secretary, and treasurer from a group of 10 persons?

(Sol.) $P_4^{10} = \dfrac{10!}{(10-4)!} = 5040$

The number of permutations around a circular table of *n* distinct objects: $(n-1)!$

Eg. In how many ways can 6 persons be seated around a circular table?

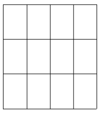
(Sol.) (6-1)!=120

Combination: Count arrangements of objects **with no reference to order**.

The number of *r*-combinations of *n* distinct objects: $C_r^n = \dfrac{n!}{r!(n-r)!}$

Eg. How many eight-bit strings contain exactly four 1's?

(Sol.) $C_4^8 = \dfrac{8!}{4!4!} = 70$



Eg. Determine the number of paths from the lower-left corner of the left grid to the upper-right corner.

(Sol.) $C_3^7 = C_4^7 = \dfrac{7!}{3!\cdot 4!} = 35$

The number of *r*-combinations of *n* distinct objects with repetition:

$C_r^{n+r-1} = \dfrac{(n+r-1)!}{r!(n-1)!}$

Eg. A fast-food restaurant provides 4 kinds of foods. They are cheese-burgers, hot dogs, sandwiches, and donuts. How many purchases are possible if 7 customers buy them?

(Sol.) n=4, r=7, $C_7^{4+7-1} = \dfrac{10!}{7!(4-1)!}$=120

Eg. In how many ways can we distribute 10 identical balls among 6 distinct boxes?

(Sol.) n=6, r=10, $C_{10}^{6+10-1} = \dfrac{15!}{10!(6-1)!}$=3003

Eg. Determine the number of all nonnegative integer solutions for x+y+z=4.

(Sol.) n=3, r=4, $C_4^{3+4-1} = \dfrac{6!}{4!(3-1)!}$=15

Eg. Determine the number of all positive integer solutions for x+y+z=6.

(Sol.) The problem is identical to "**Determine the number of all nonnegative integer solutions for x+y+z=6-3=3.**", ∴ n=3, r=3, $C_3^{3+3-1} = \dfrac{5!}{3!(3-1)!}$=10

Eg. Determine the number of all nonnegative integer solutions for x+y+z<4.

(Sol.) The problem is identical to "**Determine the number of all nonnegative integer solutions for x+y+z+w=4-1=3.**", ∴n=4, r=3, $C_3^{4+3-1} = \dfrac{6!}{3!(4-1)!}$=20

Theorem $(1+x)^n = \sum_{i=0}^{n} C_i^n x^i$ **.**

Theorem $(1+x)^{-n} = \sum_{i=0}^{\infty} C_i^{-n} x^i = \sum_{i=0}^{\infty} (-1)^i C_i^{n+i-1} x^i$ **,**

where $C_i^{-n} = \frac{(-n)!}{i!(-n-i)!} = \frac{(-n)(-n-1)(-n-2)(-n-3)\cdots(-n-i+1)}{i!} = (-1)^i C_i^{n+i-1}$ **.**

Eg. Find the coefficient of x^5 in $(1-2x)^{-7}$.

(Sol.) $(1-2x)^{-7} = \ldots + C_5^{-7}(-2x)^5 + \ldots$

$C_5^{-7} = (-1)^5 C_5^{7+5-1} = (-1)^5 C_5^{11} = -462$, $(-462)\times(-2)^5 = 14784$

Theorem **For any real number s,** $(1+x)^s = 1 + \sum_{i=0}^{\infty} \frac{s(s-1)(s-2)(s-3)\cdots(s-i+1)}{i!} x^i$ **.**

Eg. Expand $(1+3x)^{-1/3}$.

(Sol.)

$$(1+3x)^{-1/3} = 1 + \sum_{i=0}^{\infty} \frac{(\frac{-1}{3})(\frac{-1}{3}-1)(\frac{-1}{3}-2)\cdots(\frac{-1}{3}-i+1)}{i!}(3x)^i = 1 + \sum_{i=0}^{\infty} \frac{(-1)(-4)(-7)\cdots(-3i+2)}{i!} \cdot x^i \, .$$

5-2 Generating Functions

General generating function: $f(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots$ is the general generating function of a series: $a_0, a_1, a_2, a_3, \ldots$

Eg. $\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$ **is the general generating function of a series: 1, 1, 1, 1, ….**

Eg. $\frac{d}{dx}(\frac{1}{1-x})=1+2x+3x^2+4x^3+5x^4+\cdots$ **is the general generating function of a series: 1, 2, 3, 4, 5, ….**

Eg. $(1+x)^n=C_0^n+C_1^nx+C_2^nx^2+C_3^nx^3+C_4^nx^4+\cdots+C_n^nx^n$ **is the general generating function of a series:** $C_0^n,\ C_1^n,\ C_2^n,\ C_3^n,\ C_4^n,\ \cdots,\ C_n^n$**.**

Eg. $(1+x)^{-n}=C_0^{-n}+C_1^{-n}(-x)+C_2^{-n}(-x)^2+C_3^{-n}(-x)^3+C_4^{-n}(-x)^4+\cdots=\sum_{i=0}^{\infty}C_i^{n+i-1}x^i$

is the general generating function of a series: $\{C_i^{n+i-1}\}$ **,** $i \geqq 0$**.**

Eg. Find the general generating functions of (a) 1, -1, 1, -1, 1, -1, …. (b) $C_1^8,\ 2C_2^8,\ 3C_3^8,\ 4C_4^8,\ \cdots,\ 8C_8^8$**. (c) 1, 2, 4, 8, 16, …. (d) 0, 1, 2, 3, …,** n**, …. (e) 0, 1, 2, 3, …,** n**.**

(Sol.) (a) $1-x+x^2-x^3+x^4-x^5\cdots=\frac{1}{1+x}$

(b) $C_1^8+2C_2^8x+3C_3^8x^2+4C_4^8x^3+\cdots+8C_8^8x^7=(C_0^8+C_1^8x+C_2^8x^2+C_3^8x^3+\cdots+C_8^8x^8)'$

$=[(1+x)^8]'=8(1+x)^7$

(c) $1+2x+4x^2+8x^3+16x^4+\cdots=1+(2x)+(2x)^2+(2x)^3+(2x)^4+\cdots=\frac{1}{1-2x}$

(d) $x+2x^2+3x^3+4x^4+\cdots+nx^n+\cdots=\sum_{n=1}^{\infty}nx^n=x\cdot\sum_{n=0}^{\infty}(x^n)'=x\cdot(\frac{1}{1-x})'=\frac{x}{(1-x)^2}$

(e) $x+2x^2+3x^3+4x^4+\cdots+nx^n=\sum_{m=1}^{n}mx^m=x\cdot\sum_{m=0}^{n}(x^m)'=x\cdot(\frac{1-x^{n+1}}{1-x})'$

$$=x\cdot\frac{1-x^{n+1}-(n+1)(x^n-x^{n+1})}{(1-x)^2}=\frac{x-x^{n+2}-(n+1)(x^{n+1}-x^{n+2})}{(1-x)^2}$$

Eg. $f(x)=\dfrac{x-3}{x^2-3x+2}=a_0+a_1x+a_2x^2+a_3x^3+\cdots$, **$a_{50}$=?**

(Sol.) $f(x)=\dfrac{x-3}{x^2-3x+2}=\dfrac{2}{x-1}-\dfrac{1}{x-2}=\dfrac{-2}{1-x}+\dfrac{\frac{1}{2}}{1-\frac{x}{2}}=(-2)\sum_{n=0}^{\infty}x^n+\dfrac{1}{2}\sum_{n=0}^{\infty}(\dfrac{x}{2})^n$

$$a_{50}=(-2)+\frac{1}{2}(\frac{1}{2})^{50}=\frac{1}{2^{51}}-2$$

Eg. Compute $C_k^n+C_k^{n+1}+C_k^{n+2}+C_k^{n+3}+\cdots+C_k^{n+m}$.

(Sol.) $(1+x)^n+(1+x)^{n+1}+\ldots+(1+x)^{n+m}=\dfrac{(1+x)^n[1-(1+x)^{m+1}]}{1-(1+x)}=\dfrac{(1+x)^{n+m+1}-(1+x)^n}{x}$

$C_k^n+C_k^{n+1}+C_k^{n+2}+C_k^{n+3}+\cdots+C_k^{n+m}$ is a_k of Left, and $C_{k+1}^{n+m+1}-C_{k+1}^n$ is a_k of Right

$\therefore C_k^n+C_k^{n+1}+C_k^{n+2}+C_k^{n+3}+\cdots+C_k^{n+m}=C_{k+1}^{n+m+1}-C_{k+1}^n$

Eg. Compute $C_0^n+2C_1^n+4C_2^n+8C_3^n+\cdots+2^nC_n^n$.

(Sol.) Let $f(x)=(1+2x)^n=C_0^n+2xC_1^n+4x^2C_2^n+8x^3C_3^n+\cdots+(2x)^nC_n^n$

Set x=1, $C_0^n+2C_1^n+4C_2^n+8C_3^n+\cdots+2^nC_n^n=3^n$

Eg. Compute $C_1^n+2C_2^n+3C_3^n+4C_4^n+\cdots+nC_n^n$.

(Sol.) Let $f(x)=(1+x)^n=C_0^n+C_1^nx+C_2^nx^2+C_3^nx^3+C_4^nx^4+\cdots+C_n^nx^n$, then

$f'(x)=n(1+x)^{n-1}=C_1^n+2C_2^nx+3C_3^nx^2+4C_4^nx^3+\cdots+nC_n^nx^{n-1}$

Set x=1, $f'(1)=n2^{n-1}=C_1^n+2C_2^n+3C_3^n+4C_4^n+\cdots+nC_n^n$

Eg. Show that $(C_0^n)^2+(C_1^n)^2+(C_2^n)^2+(C_3^n)^2+(C_4^n)^2+\cdots+(C_n^n)^2=(C_n^{2n})$.

(Proof) Left = a_0 of $(1+x)^n(1+1/x)^n$, Right = a_0 of $\dfrac{(1+x)^{2n}}{x^n}$, $\therefore$ Left= Right.

Application of the general generating function of a series: Calculating combinations

Eg. There are 2 *A*'s and 2 *B*'s. (a) In how many ways of combinations can we select 2 letters from *A*'s, and *B*'s? (b)In how many ways can we select 3 letters from *A*'s, and *B*'s?

(Sol.) (a) *AA*, *AB* or *BA*, *BB*: 3 ways. (b) *AAB* or *BAA* or *ABA*, *BBA* or *BAB* or *ABB*: 2 ways

Utilizing the general generating function:

$f(x)=(1+x+x^2)(1+x+x^2)=1+x^2+x^4+2x+2x^2+2x^3=1+2x+3x^2+2x^3+x^4$, $a_2=3$, $a_3=2$.

Eg. There are infinite *A*'s, *B*'s, and *C*'s. In how many ways can we select *n* letters from *A*'s, *B*'s, and *C*'s, with even numbers of *A*'s?

(Sol.)

$$f(x)=(1+x^2+x^4+\cdots)(1+x+x^2+x^3+x^4+\cdots)(1+x+x^2+x^3+x^4+\cdots)$$

$$=\frac{1}{1-x^2}\cdot\frac{1}{1-x}\cdot\frac{1}{1-x}=\frac{1}{1+x}\left(\frac{1}{1-x}\right)^3=\frac{\frac{1}{8}}{1+x}+\frac{\frac{1}{8}}{1-x}+\frac{\frac{1}{4}}{(1-x)^2}+\frac{\frac{1}{2}}{(1-x)^3}$$

$$a_n=\frac{1}{8}(-1)^n+\frac{1}{8}+\frac{1}{4}(n+1)+\frac{1}{2}C_2^{n+2}$$

Eg. There are 200 identical chairs. In how many ways can we place 4 rooms to have 20, or 40, or 60, or 80, or 100 chairs in each room?

(Sol.)

$$(x^{20}+x^{40}+x^{60}+x^{80}+x^{100})^4=[x^{20}(1+x^{20}+x^{40}+x^{60}+x^{80})]^4=x^{80}\left(\frac{1-x^{100}}{1-x^{20}}\right)^4$$

$$=x^{80}(1-4x^{100}+6x^{200}-4x^{300}+x^{400})(1+4x^{20}+10x^{40}+\cdots+C_6^9x^{120}+\cdots)$$

$$a_{200}=C_6^9-16=68$$

Exponential generating function: $f(x)=b_0+b_1x+b_2\frac{x^2}{2!}+b_3\frac{x^3}{3!}+\cdots$ is the exponential generating function of a series: b_0, b_1, b_2, b_3, ….

Eg. $(1+x)^n=C_0^n+C_1^nx+C_2^nx^2+\cdots+C_n^nx^n=P_0^n+P_1^n\frac{x}{1!}+P_2^n\frac{x^2}{2!}+\cdots+P_n^n\frac{x^n}{n!}$

Applications of the exponential generating function of a series: Calculating permutations

Eg. In how many ways can we select 4 letters from *ENGINE*?

(Sol.) For *E*, *N*: we can select 0, 1, 2 but for *G*, *I*: we can select 0, 1

$$(1+x+\frac{x^2}{2!})^2(1+x)^2=(1+x^2+\frac{x^4}{(2!)^2}+2x+2\frac{x^2}{2!}+2x\frac{x^2}{2!})(1+2x+x^2)$$

$$=b_0+b_1x+b_2\frac{x^2}{2!}+b_3\frac{x^3}{3!}+b_4\frac{x^4}{4!}\cdots,\ b_4=102$$

Eg. There are 2 *A*'s and 2 *B*'s. (a) In how many ways of permutations can we select 2 letters from *A*'s, and *B*'s? (b)In how many ways can we select 3 letters from *A*'s, and *B*'s?

(Sol.) (a) *AA*, *AB*, *BA*, *BB*: 4 ways. (b) *AAB*, *BAA*, *ABA*, *BBA*, *BAB*, *ABB*: 6 ways

Utilizing the exponential generating function:

$$f(x)=(1+x+\frac{x^2}{2!})(1+x+\frac{x^2}{2!})=1+x^2+\frac{x^4}{4}+2x+x^2+x^3=1+2x+2x^2+x^3+\frac{x^4}{4}=1+\frac{2x}{1!}+\frac{4x^2}{2!}+\frac{6x^3}{3!}+\frac{6x^4}{4!},$$

$\therefore$ $b_2=4$, $b_3=6$.

Eg. A company hires 11 new employees, each of whom is to be assigned to one of 4 subdivisions. Each subdivision will get at least one new employee. In how many ways can these assignments be made?

(Sol.) $f(x)=(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots)^4=(e^x-1)^4=e^{4x}-4e^{3x}+6e^{2x}-4e^x+1$

$$=\cdots+(4^{11}-4\cdot 3^{11}+6\cdot 2^{11}-4\cdot 1^{11})\frac{x^{11}}{11!}+\cdots$$

$\therefore$ There are ($4^{11}-4\cdot 3^{11}+6\cdot 2^{11}-4$) ways!

5-3 Discrete Probability

Discrete probability functions, $P(x)$: $0 \le P(x) \le 1,\ \forall x \in S$ and $\sum_{x \in S} P(x) = 1$.

Theorem $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$.

Eg. Two fair dice are rolled. What is the probability of getting doubles (two dice showing the same number) or a sum of 6?

(Sol.) Let E_1 be get doubles and E_2 be get a sum of 6(=1+5=2+4=3+3=4+2=5+1).

$P(E_1) = 6 \cdot (\frac{1}{6})^2 = \frac{1}{6}$, $P(E_2) = 5 \cdot (\frac{1}{6})^2 = \frac{5}{36}$, $P(E_1 \cap E_2) = \frac{1}{36}$,

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{5}{18}$

Theorem If E_1 and E_2 are mutually exclusive events ($E_1 \cap E_2 = \phi$), then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Eg. Two fair dice are rolled. What is the probability of getting doubles (two dice showing the same number) or a sum of 5?

(Sol.) Let E_1 be get doubles and E_2 be get a sum of 5(=1+4=2+3=3+2=4+1).

$E_1 \cap E_2 = \phi$, $P(E_1) = 6 \cdot (\frac{1}{6})^2 = \frac{1}{6}$, $P(E_2) = 4 \cdot (\frac{1}{6})^2 = \frac{1}{9}$, $P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{5}{18}$

Theorem Let E be an event, $P(E) + P(\overline{E}) = 1$.

Theorem If E_1 and E_2 are independent events, $P(E_1 \cap E_2) = P(E_1)P(E_2)$.

Eg. Two fair coins are tossed. What is the probability of head on the first toss and tail on the second toss?

(Sol.) Let E_1 be head on the first toss and E_2 be tail on the second toss. Intuitively, E_1 and E_2 are independent events. $\therefore$ $P(E_1 \cap E_2) = P(E_1)P(E_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Conditional probability functions, $P(E|F)$: $P(E|F) = P(E \cap F)/P(F)$

Eg. There was a test for AIDS with the property that 90% of those with AIDS reacted positively whereas 5% of those without AIDS react positively. Assume that 1% of patients in a hospital have AIDS. What is the probability that a patient selected at random who reacts positively to this test actually has AIDS?

(Sol.) P(AIDS|React positively)= P(AIDS∩React positively)/P(React positively)

$$= \frac{\frac{1}{100} \times \frac{90}{100}}{\frac{1}{100} \times \frac{90}{100} + \frac{99}{100} \times \frac{5}{100}} = \frac{2}{13}$$

Bayes' Theorem **Suppose that the mutual exclusive classes are C_1, C_2, C_3, …, and C_n. For a feature set F, we have** $P(C_j|F) = \dfrac{P(F|C_j)P(C_j)}{\sum_{i=1}^{n} P(F|C_i)P(C_i)}$.